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# The cyclic structure of unimodal permutations

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## Abstract

Unimodal (i.e. single-humped) permutations may be decomposed into a product of disjoint cycles. Some enumerative results concerning their cyclic structure — e.g.  $\frac{2}{3}$  of them contain fixed points — are given. We also obtain in effect a kind of combinatorial universality for continuous unimodal maps, by severely constraining the possible ways periodic orbits of any such map can nestle together. But our main observation (and tool) is the existence of a natural noncommutative monoidal structure on this class of permutations which respects their cyclic structure. This monoidal structure is a little mysterious, and can perhaps be understood by broadening the context, e.g. by looking for similar structure in other classes of ‘pattern-avoiding’ permutations.

## 1. Introduction

Let  $\Delta(n)$  denote the set of all unimodal permutations  $\delta$  of  $I_n := \{1, 2, \dots, n\}$ . That is, for any such  $\delta \in \Delta(n)$  there exists an  $m \in I_n$  satisfying

- (i)  $a < b \leq m \Rightarrow \delta(a) < \delta(b)$ , and
- (ii)  $m \leq a < b \Rightarrow \delta(a) > \delta(b)$ .

Of course,  $m = \delta^{-1}(n)$  is the maximum. Write  $\Delta(\star)$  for  $\cup_n \Delta(n)$ , and  $\mathfrak{S}_n$  for the symmetric group.

$\Delta(\star)$  is the discrete analogue of the unimodal maps studied in 1-dimensional dynamical systems (see e.g. the classic [3]). For instance, it has been observed that small populations have a tendency to grow, and large ones decrease. The simplest model for this is a unimodal function. This was the motivation presented in [6] for analysing the cyclic structure of  $\delta \in \Delta(\star)$ . More generally, if  $f$  is any continuous unimodal map and  $J$  is any finite set for which  $f(J) = J$ , then the restriction  $f|_J$  is ‘topologically conjugate’ to a unique  $\delta \in \Delta(\star)$ , called the ‘permutation type’ of  $f|_J$  (explicitly,  $\delta = \Omega_J^{-1} \circ f \circ \Omega_J$  where  $\Omega_J$  is an increasing bijection to be defined shortly). In contrast to that of periodic orbits, the theory of finite invariant sets  $J$  of maps  $f : I \rightarrow I$  is still largely undeveloped — a notable exception is [5] — and this paper can be regarded as a move in that direction for the special case of unimodal maps. We return to this context in section 3.

Unimodal permutations also appear naturally in a second context. We say that a permutation  $\pi \in \mathfrak{S}_n$  ‘contains’ a pattern  $\sigma = [\sigma_1, \sigma_2, \dots, \sigma_k] \in \mathfrak{S}_k$  if there exist  $k$  indices  $1 \leq i_{\sigma_1} < i_{\sigma_2} < \dots < i_{\sigma_k} \leq n$  such that  $\pi(i_1) < \pi(i_2) < \dots < \pi(i_k)$ , otherwise we say  $\pi$  ‘avoids’  $\sigma$  [7]. Equivalently,  $\sigma$  is contained in  $\pi$  iff the permutation matrix of  $\sigma$  is a submatrix of the permutation matrix of  $\pi$ . For example,  $[3, 2, 4, 1]$  contains the patterns  $[2, 1, 3]$  (take the 3 indices  $\{1, 2, 3\}$ ) and  $[2, 3, 1]$  (take e.g. indices  $\{1, 3, 4\}$ ), but avoids  $[1, 2, 3]$  and  $[3, 1, 2]$ . Write  $\mathfrak{S}_n(\sigma_1, \sigma_2, \dots)$  for the set of all  $\pi \in \mathfrak{S}_n$  avoiding all  $\sigma_i$ . Questions involving pattern-avoidance arise for example in sorting problems in computing science. A slightly more general notion: call a set  $S \subseteq \cup_n \mathfrak{S}_n$  of permutations ‘closed’ [1] if for any pattern  $\sigma$  contained in any  $\pi \in S$ , we have  $\sigma \in S$ . Now,  $\Delta(n)$  is precisely the set  $\mathfrak{S}_n([213], [312])$  of all those permutations which avoid both patterns  $[2, 1, 3]$  and  $[3, 1, 2]$ , and  $\Delta(\star)$  is closed.

It is easy to calculate the cardinality of  $\Delta(n)$ . Note that

$$\|\{\delta \in \Delta(n) \mid \delta^{-1}(n) = m\}\| = \binom{n-1}{m-1},$$

so we get  $\|\Delta(n)\| = 2^{n-1}$ . For example, the four permutations in  $\Delta(3)$  are  $[231] = (123)$ ,  $[132] = (1)(23)$ ,  $[321] = (13)(2)$  and  $[123] = (1)(2)(3)$ .

Considerably more difficult is the determination of the cardinality of the *transitive* unimodal permutations — the *n-cycles* —, the set of which we will denote  $\Delta_n$ . For example,  $\Delta_5$  consists of the cycles  $(12345)$ ,  $(13425)$  and  $(12435)$ . Weiss and Rogers [8], using methods related to [4], obtained

$$\|\Delta_n\| = \frac{1}{n} \sum_{\substack{d|n \\ d \text{ odd}}} \mu(d) 2^{\frac{n}{d}-1}, \quad (1)$$

where  $\mu$  is the Möbius function. Thus about  $\frac{1}{n}$  of the permutations in  $\Delta(n)$  are transitive (of course,  $\frac{1}{n}$  is also the corresponding fraction for  $\mathfrak{S}_n$ ). The formula in (1) appears in other contexts: for instance, it counts the number of bifurcations of stable periodic orbits of the quadratic family  $x \mapsto x^2 - a$ . Write  $\Delta_\star := \cup_k \Delta_k$ .

Let  $J$  be any subset of  $\mathbb{R}$  with cardinality  $k$ . Define  $\Omega_J : I_k \rightarrow J$  to be the unique increasing bijection from  $I_k$  to  $J$ . We are mostly interested in  $J \subset \mathbb{N} := \{1, 2, \dots\}$ , in which case put  $\overline{\Omega}_J = \Omega_{\mathbb{N} \setminus J} : \mathbb{N} \rightarrow \mathbb{N} \setminus J$ .

Any  $\delta \in \Delta(n)$  decomposes uniquely of course into a product of pairwise disjoint cycles. Each cycle will also be unimodal: if  $\delta|_J$  is a cycle, then  $\delta_J := \Omega_J^{-1} \circ \delta \circ \Omega_J \in \Delta_{\|J\|}$ . We shall call  $\delta_J$  the *shape*, and  $\|J\|$  the *length*, of  $\delta|_J$ . Cycles of length 1 are fixed-points.

In this paper we will investigate questions concerning the cyclic structure of unimodal permutations — see e.g. equations (2),(7) below. We shall find, for example, that  $\frac{2}{3}$  of all unimodal permutations have fixed-points and  $\frac{2}{5}$  have 2-cycles, compared with  $1 - e^{-1}$  and  $1 - \sqrt{e}^{-1}$  of all permutations, respectively. It will also be found that many combinatorial properties of a cycle are independent of its shape.

Given any  $\delta$ , write  $N_\delta : \Delta_\star \rightarrow \{0, 1, 2, \dots\}$  for the *cycle-counter*, where  $N_\delta(\delta')$  equals the number of cycles in  $\delta$  with shape  $\delta'$ . For example  $N_{(1)(26)(35)(4)}(12) = 2$ . Note that for any  $\delta \in \Delta(n)$ ,  $n = \sum_k \sum_{\delta' \in \Delta_k} k N_\delta(\delta')$ .

A complementary question to computing  $\|\Delta_k\|$  is, what is the number  $\mathcal{N}(N)$  of  $\delta$  with a given cycle-counter  $N_\delta = N$ ? This question (actually a less fundamental one involving only the lengths and not shapes of subcycles) was asked in [6]. The answer turns out to be simple:

$$\mathcal{N}(N) = 2^{\ell-1} \quad (2)$$

where  $\ell$  is the number of distinct  $\delta' \in \Delta_\star$  with  $N(\delta') \neq 0$ . Such a simple answer should be hinting at some deeper structure. Indeed, our proof of (2) will be constructive, in that we will find an associative but noncommutative operation ‘ $\boxplus$ ’ from (a subset of)  $\Delta(n) \times \Delta(n')$  onto  $\Delta(n + n')$ , obeying

$$N_{\delta \boxplus \delta'} = N_\delta + N_{\delta'} . \quad (3)$$

Then (2) is essentially the statement that every  $\delta \in \Delta(\star)$  can be built up uniquely from  $\boxplus$ .

Similar questions can be asked for other pattern-avoiding classes of permutations. Standard practise in combinatorics is to enumerate certain sets, and when two sets are discovered to have the same cardinality, to try to establish an explicit bijection between them. Not surprisingly, the focus here has been on enumeration questions, although [1] has called for a structure theory of ‘closed sets’. For instance, Knuth (1973) showed that for any  $\sigma \in \mathfrak{S}_3$ , the cardinality  $\|\mathfrak{S}_n(\sigma)\|$  equals the  $n$ th Catalan number, while [7] showed  $\|\mathfrak{S}_n([123], [132], [213])\|$  is the  $(n + 1)$ -th Fibonacci number. Curiously, questions of cyclic structure appear to have been ignored, and yet pattern-avoiding classes of permutations are precisely those classes for which cyclic structure is natural to investigate — that is, their subcycles avoid those same patterns. We see in this paper that for at least some such classes, e.g. the unimodal ones, we get interesting answers. We briefly return to this in section 3.

Similarly, one could hope that other closed sets of permutations would have a nice monoidal structure. Another obvious direction is to try to extend this theory to multimodal permutations. Also interesting should be (unimodal) *nonbijective* functions  $\gamma : I_n \rightarrow I_n$  — these are considered e.g. in [5,6].

## 2. The monoidal structure

Consider any  $\delta_1 \in \Delta(k)$ ,  $\delta_2 \in \Delta(\ell)$ , and put  $m_1 = \delta_1^{-1}(k)$  and  $m_2 = \delta_2^{-1}(\ell)$ . Choose any  $J \subset I_{k+\ell}$  with  $\|J\| = k$  and write  $\Omega$  for  $\Omega_J$ ,  $\overline{\Omega}$  for  $\overline{\Omega}_J$ .

By the ‘sum’  $\delta_1 \oplus_J \delta_2$  (or just  $\delta_1 \oplus \delta_2$  if  $J$  is understood) we mean the permutation satisfying, for all  $i \in I_{k+\ell}$ ,

$$(\delta_1 \oplus_J \delta_2)(i) = \begin{cases} (\Omega \circ \delta_1 \circ \Omega^{-1})(i) & \text{if } i \in J \\ (\overline{\Omega} \circ \delta_2 \circ \overline{\Omega}^{-1})(i) & \text{if } i \notin J \end{cases}. \quad (4)$$

That is,  $\delta_1$  and  $\delta_2$  are intertwined, with  $\delta_1$  placed at  $J$ . For example,  $(13425) \oplus_{\{2,4,5,6,7\}} (123) = (138)(25647)$ .

Obviously  $N_{\delta_1 \oplus \delta_2} = N_{\delta_1} + N_{\delta_2}$ . Of course,  $m := (\delta_1 \oplus \delta_2)^{-1}(k+\ell) \in \{\Omega(m_1), \overline{\Omega}(m_2)\}$ . This operation ‘ $\oplus$ ’ obeys a kind of commutativity and associativity: e.g.  $\delta_1 \oplus_J \delta_2 = \delta_2 \oplus_{I_{k+\ell} \setminus J} \delta_1$ . Of course any  $\delta \in \Delta(\star)$  can be written as  $\delta = \delta' \oplus_{J'} \delta'' \oplus_{J''} \cdots \oplus_{J^{(s-1)}} \delta^{(s)}$  using obvious notation, where  $\delta^{(i)}$  is the shape of the subcycle  $\delta|_{J^{(i)}}$  of  $\delta$ .

Our immediate problem is, given  $\delta_1$  and  $\delta_2$ , to find all  $J$  such that  $\delta_1 \oplus_J \delta_2$  is unimodal. Our goal is Theorem 6. Without loss of generality, we will assume unless otherwise stated that  $J$  obeys  $\Omega(m_1) < \overline{\Omega}(m_2)$ . The following result follows trivially from unimodality, and hints at how  $\delta_1$  and  $\delta_2$  must fit together.

**Lemma 1.** *Assume  $\delta_1 \oplus \delta_2 \in \Delta(k+\ell)$ . Then for any  $a \in I_k$ ,  $b \in I_\ell$ ,*

- $a > m_1$  and  $b \geq m_2 \Rightarrow \Omega(a) > m$  and  $\overline{\Omega}(b) \geq m$ ;
- $a \leq m_1$  and  $b < m_2 \Rightarrow \Omega(a) \leq m$  and  $\overline{\Omega}(b) < m$ ;
- $a > m_1$  and  $b < m_2 \Rightarrow \Omega(a) > \overline{\Omega}(b)$ ;
- $a \leq m_1$  and  $b \geq m_2 \Rightarrow \Omega(a) < \overline{\Omega}(b)$ .

Simple as it is, this Lemma provides a major clue to the ideas which follow. Indeed, partition the pairs  $I_k \times I_\ell$  into 4 sets  $\mathcal{P}_{>\geq}$ ,  $\mathcal{P}_{\leq<}$ ,  $\mathcal{P}_{><}$  and  $\mathcal{P}_{\leq\geq}$  defined in the obvious way (e.g.  $\mathcal{P}_{>\geq}$  consists of all pairs  $(a, b)$  where  $a > m_1$  and  $b \geq m_2$ ). Our approach is related to that of [4], except in how we must treat the ‘turning points’  $m_i$ .

Specialise for now to  $\delta_1 \in \Delta_k$ ,  $\delta_2 \in \Delta_\ell$ , and choose any  $a \in I_k$ ,  $b \in I_\ell$ . Define  $a_i := \delta_1^i(a)$ ,  $b_i := \delta_2^i(b)$  for all  $i \geq 0$ . We shall consider the successive iterates  $(a_0, b_0)$ ,  $(a_1, b_1)$ , etc., up to the smallest  $L$  for which  $(a_L, b_L) \in \mathcal{P}_{><} \cup \mathcal{P}_{\leq\geq}$  (such an  $L$  always exists by Lemma 2 below). From Lemma 1 we then know the relative ordering of  $\Omega(a_L)$  and  $\overline{\Omega}(b_L)$ . Now going backwards, we can use unimodality (via Lemma 1) to determine the ordering of  $\Omega(a_{L-1})$  and  $\overline{\Omega}(b_{L-1})$ , then  $\Omega(a_{L-2})$  and  $\overline{\Omega}(b_{L-2})$ , and ultimately  $\Omega(a_0)$  and  $\overline{\Omega}(b_0)$ . In this way we can (indirectly) find the unique set  $J$ , and hence the unique unimodal sum  $\delta_1 \oplus_J \delta_2$ . The next several lines, culminating in Theorem 3, fill in this sketch.

Let  $S = S(a, b)$  be the sequence whose  $i$ th term  $S_i$  is 1, 2, 3, or 4 if  $(a_i, b_i) \in \mathcal{P}_{>\geq}$ ,  $\mathcal{P}_{\leq<}$ ,  $\mathcal{P}_{><}$  or  $\mathcal{P}_{\leq\geq}$ , respectively. Call this sequence ‘finite of length  $L \geq 0$ ’ if  $S_i \in \{1, 2\}$  for all  $0 \leq i < L$  and  $S_L \in \{3, 4\}$ .

**Lemma 2.**  $S(a, b)$  is always finite.

**Proof.** Suppose for contradiction that for each  $i$ ,  $a_i \leq m_1$  iff  $b_i < m_2$ . Then clearly both  $k, \ell > 1$ . Without loss of generality take  $a_0 = m_1$  (so  $a_1 = k$  and  $b_0 < m_2$ ), and let  $n > 0$  satisfy  $b_n = \ell$ . Then if  $a_n = k$ , we would have  $a_{n-1} = m_1$  and  $b_{n-1} = m_2$ , contradicting our supposition.

Therefore our supposition yields both  $a_1 > a_n > m_1$  and  $b_n > b_1 \geq m_2$ . Thus  $a_{1+1} < a_{n+1}$  and  $b_{1+1} > b_{n+1}$ , which requires  $S_{1+1} = S_{n+1}$ . This in turn implies  $a_{1+2} < a_{n+2}$  iff  $b_{1+2} > b_{n+2}$ , etc. Inductively, we get that  $S$  has period  $n - 1$ :  $S_{1+i} = S_{n+i}$ . But then  $S_{k\ell+n-1} = S_{n-1} = 1$  contradicts  $S_{k\ell} = S_0 = 2$ .  $\square$

Let  $S(a, b)$  be of length  $L$ , and let  $M$  be the number of  $\ell \leq L$  such that  $S_\ell$  is 1 or 3. Write  $a \succ b$  if  $M$  is odd, otherwise  $a \prec b$ .

**Theorem 3.** Given  $\delta_1 \in \Delta_k$  and  $\delta_2 \in \Delta_\ell$ , there exists exactly one set  $J$  with  $\delta_1 \oplus_J \delta_2 \in \Delta(k + \ell)$ , satisfying  $\Omega(m_1) < \overline{\Omega}(m_2)$ .

**Proof.** By Lemma 1, if we have  $\delta_1 \oplus_J \delta_2 \in \Delta(k + \ell)$ , then we immediately get:

$$\Omega_J(a) < \overline{\Omega}_J(b) \iff a \prec b. \quad (5)$$

There is at most one cardinality- $k$  set  $J \subset I_{k+\ell}$  which obeys (5). Conversely, by the definition of  $\prec$  and  $\succ$ , if we find such a set  $J$ , then  $\delta_1 \oplus_J \delta_2$  will necessarily be unimodal with  $\Omega_J(m_1) < \overline{\Omega}_J(m_2)$ . Such a set will exist if, for any  $a \in I_k, b \in I_\ell$ , we have

- (i) if  $c \in I_k, c < a$  and  $a \prec b$ , then  $c \prec b$ ;
- (ii) if  $c \in I_\ell, c > b$  and  $a \prec b$ , then  $a \prec c$ ;
- (iii) if  $c \in I_k, c > a$  and  $a \succ b$ , then  $c \succ b$ ;
- (iv) if  $c \in I_\ell, c < b$  and  $a \succ b$ , then  $a \succ c$ .

The proof of (i)-(iv) is a simplified version of the proof of Proposition 5 given below.  $\square$

Of course, by ‘commutativity’ of  $\oplus$ , we also get that there is exactly one  $J'$  for which  $\delta_1 \oplus_{J'} \delta_2 \in \Delta(k + \ell)$  and  $\Omega_{J'}(m_1) > \overline{\Omega}_{J'}(m_2)$ . Thus for transitive  $\delta_1 \neq \delta_2$ , there are precisely 2 distinct  $\delta \in \Delta(k + \ell)$  with cycles of shape  $\delta_1, \delta_2$ ; while for  $\delta_1 = \delta_2$  there will be exactly 1 such  $\delta$ . An important relation between the two  $\delta_1 \oplus \delta_2$  is given in Proposition 5(c) below.

For example, let  $\delta_1 = (123)$ ,  $\delta_2 = (13425)$ . Then the sums of the form  $\delta_1 \oplus \delta_2$ ,  $\delta_1 \oplus \delta_1$ ,  $\delta_2 \oplus \delta_2$  are  $(138)(25647)$  (for  $m_1 \prec m_2$ ) and  $(148)(25637)$  (for  $m_1 \succ m_2$ ),  $(136)(245)$ , and  $(158310)(26749)$ . These have  $J = \{1, 3, 8\}$ ,  $\{1, 4, 8\}$ ,  $\{1, 3, 6\}$  and  $\{1, 3, 5, 8, 10\}$ , respectively.

**Corollary 4.** Let  $\delta \in \Delta_n$ , and write  $m = \delta^{-1}(n)$ . Let  $J$  be the set in Theorem 3. Then  $J$  contains exactly one element from  $\{1, 2\}$ , one from  $\{3, 4\}$ , etc. Moreover,  $(\delta \oplus \delta)^{-1}(2n) = 2m$  if  $n \equiv m \pmod{2}$ ; otherwise  $(\delta \oplus \delta)^{-1}(2n) = 2m - 1$ .

Similar comments hold for repeated sums  $\delta \oplus \cdots \oplus \delta$ . To see the first assertion in Corollary 4, apply unimodality repeatedly to the inequalities  $\ell \succ \ell + 1$  and  $\ell < \ell + 1$  in

order to produce a contradiction. The second assertion follows by counting the number of times  $\delta^\ell(1) > m$  for  $1 < \ell < n - 2$  ( $n - 2$  is the length of  $S(1, 1)$ ), to determine whether or not  $1 \succ 1$ .

The following technical definition is crucial.

**Definition.** Call  $\delta' \in \Delta_k$  acute if  $n \equiv \delta'^{-1}(k) \pmod{2}$  (so the two maxima of  $\delta' \oplus \delta'$  run diagonally SW–NE like ‘/’), otherwise call it grave. Choose any  $\delta \in \Delta(n)$  and let  $\delta(J) = J$ , and write  $m(\delta|_J)$  for the maximum of  $\delta|_J$ . By  $\mathcal{A}(\delta)$  we mean the set of all acute  $\delta' \in \Delta_*$  which are the shapes of subcycles  $\delta|_J$  of  $\delta$ ; similarly,  $\mathcal{G}(\delta)$  will be the grave shapes in  $\delta$ . Write  $\delta' \in \mathcal{A}_>(\delta)$  if  $\delta' \in \mathcal{A}(\delta)$  and there is some subset  $J \subset I_n$  such that  $\delta|_J$  has shape  $\delta'$  and  $m(\delta|_J) > m(\delta)$ ; define  $\mathcal{A}_<(\delta), \mathcal{G}_>(\delta), \mathcal{G}_<(\delta)$  similarly.

For example,  $(12 \dots k)$  is acute iff  $k = 1$ . For  $\delta = (1)(26)(35)(4)$ ,  $\mathcal{A}_>(\delta) = \mathcal{A}_<(\delta) = \{(1)\}$ ,  $\mathcal{G}_>(\delta) = \{(12)\}$  and  $\mathcal{G}_<(\delta) = \emptyset$ .

$\mathcal{A}_<(\delta)$  and  $\mathcal{G}_>(\delta)$  can be thought of as the subcycles of  $\delta$  of ‘positive type’ [4] (or ‘orientation-preserving’ subcycles). Put another way, think of  $\mathcal{A}_<(\delta), \dots, \mathcal{G}_>(\delta)$  as *multi-sets*, i.e. their elements come with multiplicity. Then we will find that the multiplicities in  $\mathcal{A}_<$  and  $\mathcal{G}_>$  can be arbitrarily large, but those of  $\mathcal{A}_>$  and  $\mathcal{G}_<$  can never exceed 1. Because of this,  $\mathcal{A}_>$  and  $\mathcal{G}_<$  will play an important role in Theorem 6 below.

We must generalise Theorem 3 by removing the transitivity requirement. This is equivalent to considering multiple sums.

Select any  $\delta_i \in \Delta_{n_i}$ , for  $i = 1, 2, 3$ . We are interested in constructing unimodal sums  $\delta = \delta_1 \oplus_{J_1} \delta_2 \oplus_{J_2} \delta_3$  of these three permutations which obey  $\Omega_1(m_1) < \Omega_2(m_2) < \Omega_3(m_3)$ , by applying the preceding analysis to the partial sums  $\delta_{ij} := \delta_i \oplus \delta_j$ . Here and elsewhere we write  $\Omega_i$  for  $\Omega_{J_i}$ , and  $m_i = \delta_i^{-1}(n_i)$ . Define  $\prec_{ij}, \succ_{ij}$  for  $\delta_{ij}$  as before. We will require as usual that each  $m_i \prec_{ij} m_j$ . Note that we have no hope to construct a unimodal sum  $\delta_1 \oplus \delta_2 \oplus \delta_3$  with  $\Omega_1(m_1) < \Omega_2(m_2) < \Omega_3(m_3)$ , if both  $n_1 \succ_{12} n_2$  and  $n_2 \prec_{23} n_3$ . We will find that this is the only obstacle; to show that, we must establish the compatibility of the orderings  $\prec_{ij}$ .

**Proposition 5.** Choose any  $\delta_i \in \Delta_{n_i}$ , and let  $\delta_{ij}$  be as in the preceding paragraph.

- (a) Assume that either  $n_1 \prec_{12} n_2$  or  $n_2 \succ_{23} n_3$ . Then for any  $a \in I_{n_1}$ ,  $b \in I_{n_2}$ ,  $c \in I_{n_3}$ , we have both
  - (i)  $a \prec_{12} b$  and  $b \prec_{23} c$  implies  $a \prec_{13} c$ ;
  - (ii)  $a \succ_{12} b$  and  $b \succ_{23} c$  implies  $a \succ_{13} c$ .
- (b) There exist sets  $J_i$  such that  $\Omega_1(m_1) < \Omega_2(m_2) < \Omega_3(m_3)$  and  $\delta_1 \oplus_{J_1} \delta_2 \oplus_{J_2} \delta_3$  is unimodal, iff either  $n_1 \prec_{12} n_2$  or  $n_2 \succ_{23} n_3$ . Moreover, when such sets  $J_i$  exist, they will be unique.
- (c) Suppose  $\delta \in \Delta_k$ ,  $\delta' \in \Delta_\ell$ ,  $\delta \neq \delta'$ , and let  $\delta \oplus_{J_A} \delta'$  and  $\delta \oplus_{J_B} \delta'$  be the two distinct unimodal sums. Then  $k \prec_A \ell$  iff  $k \prec_B \ell$ .

**Proof of (a).** Assume for contradiction that we have found  $a, b, c$  so that  $a \prec_{12} b$  iff  $b \prec_{23} c$  iff  $a \succ_{13} c$ . Write  $a_\ell = \delta_1^\ell(a)$ ,  $b_\ell = \delta_2^\ell(b)$ ,  $c_\ell = \delta_3^\ell(c)$ ,  $m_{ij} = \delta_{ij}^{-1}(n_i + n_j)$ , and let  $\Omega_{ij} : I_{n_i} \rightarrow I_{n_i + n_j}$ ,  $\overline{\Omega}_{ij} : I_{n_j} \rightarrow I_{n_i + n_j}$  be the increasing maps which build up  $\delta_{ij}$ .

Put  $L_{13}$  for the length of the sequence  $S_{13}(a, c)$  — i.e. the smallest  $0 \leq \ell < \infty$  such that  $(a_\ell, c_\ell) \in \mathcal{P}_{><} \cup \mathcal{P}_{\leq \geq}$ . Define  $L'_{12}$  to be the smallest  $0 \leq \ell \leq \infty$  such that either

$\Omega_{12}(a_\ell) > m_{12} > \overline{\Omega}_{12}(b_\ell)$  or  $\Omega_{12}(a_\ell) < m_{12} < \overline{\Omega}_{12}(b_\ell)$ . Define  $L'_{23}$  similarly.  $L'_{ij}$  is the furthest point to which we can carry a recursive unimodality argument for  $\delta_{ij}$ .

Let  $L = \min\{L_{13}, L'_{12}, L'_{23}\} < \infty$ . For each  $\ell < L$ , we get by definition either:

- $\Omega_{12}(a_\ell) \leq m_{12}$ ,  $\overline{\Omega}_{12}(b_\ell) \leq m_{12}$ ,  $\Omega_{23}(b_\ell) \leq m_{23}$ ,  $\overline{\Omega}_{23}(c_\ell) \leq m_{23}$ ,  $\Omega_{13}(a_\ell) \leq m_{13}$ ,  $\overline{\Omega}_{13}(c_\ell) \leq m_{13}$ ; or
- $\Omega_{12}(a_\ell) \geq m_{12}$ ,  $\overline{\Omega}_{12}(b_\ell) \geq m_{12}$ ,  $\Omega_{23}(b_\ell) \geq m_{23}$ ,  $\overline{\Omega}_{23}(c_\ell) \geq m_{23}$ ,  $\Omega_{13}(a_\ell) \geq m_{13}$ ,  $\overline{\Omega}_{13}(c_\ell) \geq m_{13}$ .

Therefore unimodality repeatedly applied to “ $a_1 \prec_{12} b_1$  iff  $b_1 \prec_{23} c_1$  iff  $a_1 \succ_{13} c_1$ ” yields

$$a_L \prec_{12} b_L \text{ iff } b_L \prec_{23} c_L \text{ iff } a_L \succ_{13} c_L. \quad (6)$$

**Case i.**  $L = L'_{12} < L_{13}$ .

$L < L_{13}$  means  $a_L > m_1$  iff  $c_L \geq m_3$ . The  $L = L'_{12}$  condition implies  $a_L > m_1$  iff  $b_L < m_2$  iff  $a_L \succ_{12} b_L$ . Putting all this together with (6) forces  $b_L = m_2$  and  $n_2 \prec_{23} n_3$ , hence  $n_1 \prec_{12} n_2$ , which contradicts  $L = L'_{12}$ .

**Case ii.**  $L = L'_{23} < L_{13}$ .

This is handled identically to **Case i**.

**Case iii.**  $L = L_{13}$ .

$L = L_{13}$  means  $a_L > m_1$  iff  $c_L < m_3$ , iff  $a_L \succ_{13} c_L$  iff  $b_L \prec_{23} c_L$  iff  $a_L \prec_{12} b_L$ . This forces  $a_L > m_1$ ,  $c_L < m_3$ ,  $b_L = m_2$ ,  $b_L \prec_{23} c_L$  and  $a_L \prec_{12} b_L$ , and hence both  $n_1 \succ_{12} n_2$  and  $n_2 \prec_{23} n_3$ , contrary to hypothesis.

**Proof of (b).** Immediate from (a).

**Proof of (c).** Without loss of generality take  $k \geq \ell$ , and suppose for contradiction  $k \prec_A \ell$  but  $k \succ_B \ell$ . Then by part (b), there exist sets  $J_i$  such that  $\gamma := \delta \oplus_{J_1} \delta' \oplus_{J_2} \delta$  is unimodal and  $\Omega_1(m) < \Omega_2(m') < \Omega_3(m)$ . Write  $a_\ell = \gamma^\ell(\Omega_1(m))$ ,  $b_\ell = \gamma^\ell(\Omega_2(m'))$ ,  $c_\ell = \gamma^\ell(\Omega_3(m))$ . The result follows from Corollary 4 and repeated unimodality: for each  $\ell$  we get either  $a_\ell < b_\ell < c_\ell$  or  $a_\ell > b_\ell > c_\ell$ , hence  $k = \ell$  and  $\delta = \delta'$ .  $\square$

For any  $\delta \in \Delta_k, \delta' \in \Delta_\ell$ , write  $\delta \triangleleft \delta'$  if  $\delta \neq \delta'$  and  $k \prec \ell$  in  $\delta \oplus \delta'$ . Proposition 5 tells us that this gives us a total-ordering on  $\Delta_\star$ . The 1-cycle (1) is the minimal element, (12) is the second smallest, and there is no maximal element: in fact  $\delta \triangleleft (12 \dots n)$  for any  $\delta \in \Delta_k$ ,  $\delta \neq (12 \dots n)$ , with  $\delta^{-1}(k) < n$ . In fact this is precisely the ordering on  $\Delta_\star$  discussed by Metropolis-Stein-Stein (1973), and extended into a refinement of the Sarkovskii ordering  $3 >_s 5 >_s \dots >_s 8 >_s 4 >_s 2 >_s 1$  of  $\mathbb{N}$ , by Baldwin et al (see [2,5] and references therein). In particular,  $\delta \triangleleft \delta'$  iff any continuous map  $f : I \rightarrow I$  having a periodic orbit with permutation type  $\delta'$  will necessarily have another with type  $\delta$  (fix  $I = [0, 1]$ , say). In [2] this is extended to arbitrary (i.e. nonunimodal) cycles, where the ordering (called ‘forcing’) is partial, and in [5] forcing is further extended to arbitrary maps  $\gamma : I_n \rightarrow I_n$ , where it is no longer antisymmetric. In the unimodal case, everything is simpler. Write  $\delta_{(k)}$  for  $\min \Delta_k$ ; e.g. for odd  $k$ ,  $\delta_{(k)} = (1, \frac{n+1}{2}, \frac{n+1}{2} + 1, \frac{n+1}{2} - 1, \dots, \frac{n+1}{2} - \frac{n-3}{2}, \frac{n+1}{2} + \frac{n-1}{2})$ . Then  $k <_s \ell$  iff  $\delta_{(k)} \triangleleft \delta_{(\ell)}$ . A theorem of Bernhardt (1987), or our Corollary 4, implies that a given  $\delta \in \Delta_\star$  has an immediate predecessor  $\delta'$  (with respect to ‘ $\triangleleft$ ’) iff  $\delta$  is the ‘double’  $\mathcal{D}\delta'$  of  $\delta'$  (see the end of this section). These comments on  $\triangleleft$  are not used in this paper.

We are now prepared for the general theorem on  $\oplus$ .

**Theorem 6.** Let  $\delta_i \in \Delta(n_i)$ ,  $i = 1, 2$ . Define  $m_i = \delta_i^{-1}(n_i)$ ,  $J_i = \{1, \delta_i(1), \delta_i^2(1), \dots\}$ , and  $\widehat{\delta}_i = \Omega_i^{-1} \circ \delta_i \circ \Omega_i$ . Then:

- (i) if either  $\mathcal{A}_>(\delta_1) \cap \mathcal{A}_>(\delta_2)$  or  $\mathcal{G}_<(\delta_1) \cap \mathcal{G}_<(\delta_2)$  are nonempty, then there are no unimodal sums of the form  $\delta_1 \oplus \delta_2$ ;
- (ii) if instead  $\widehat{\delta}_1 \in \mathcal{G}_<(\delta_2)$  or  $\widehat{\delta}_2 \in \mathcal{A}_>(\delta_1)$ , then there is no unimodal sum  $\delta_1 \oplus \delta_2$  with  $m_1 \prec m_2$ ;
- (iii) otherwise, there is exactly one unimodal sum  $\delta_1 \oplus \delta_2$  with  $m_1 \prec m_2$ .

The proof follows from repeated application of Proposition 5(b).  $\widehat{\delta}_i$  is the shape of the subcycle of  $\delta_i$  containing the maximum. Of course the analogous statements to those in Theorem 6(ii),(iii) hold for unimodal sums  $\delta_1 \oplus \delta_2$  with  $m_1 \succ m_2$ .

**Definition.** Given  $\delta_i \in \Delta(n_i)$ , denote by  $\delta_1 \boxplus \delta_2$  the unique unimodal sum  $\delta_1 \oplus_J \delta_2$  obeying  $m_1 \prec m_2$  (when it exists).

We thus get a (partial) monoidal structure on  $\Delta(\star)$ . It is associative but not commutative:

**Proposition 7.** Let  $\delta_i \in \Delta(\star)$ .

- (a) If both  $\delta_1 \boxplus \delta_2$  and  $\delta_2 \boxplus \delta_1$  exist, then they will be equal iff  $\widehat{\delta}_1 = \widehat{\delta}_2$ , using the notation of Theorem 6.
- (b) If  $\delta_1 \boxplus (\delta_2 \boxplus \delta_3)$  exists, then so does  $(\delta_1 \boxplus \delta_2) \boxplus \delta_3$  and they are equal.

We can extend the domain of ‘ $\boxplus$ ’ to all of  $\Delta(\star) \times \Delta(\star)$ , in the following natural way. Define the *double*  $\mathcal{D}\delta \in \Delta_{2k}$  of  $\delta \in \Delta_k$  to be

$$(\mathcal{D}\delta)(i) = \begin{cases} (\delta \boxplus \delta)(i) & \text{if } i \notin \{2m-1, 2m\} \\ (\delta \boxplus \delta)(4m-1-i) & \text{otherwise} \end{cases}.$$

For example,  $\mathcal{D}(12 \dots k) = (1, 3, \dots, 2k-1, 2, 4, \dots, 2k)$ . It is a consequence of Corollary 4 that for any  $\delta \in \Delta_\star$ ,  $\mathcal{D}\delta$  is the immediate successor of  $\delta$ , and that  $\delta$  is acute iff  $\mathcal{D}\delta$  is grave.

Now, for any  $\delta \in \Delta(n)$  and  $\delta' \in \Delta_k$ , define  $\delta \boxplus_e \delta' \in \Delta(n+k)$  by

$$\delta \boxplus_e \delta' = \begin{cases} \delta \boxplus \delta' & \text{if } \delta' \notin \mathcal{A}_>(\delta) \\ (\overline{\Omega}_J^{-1} \circ \delta \circ \overline{\Omega}_J) \boxplus \mathcal{D}\delta' & \text{otherwise} \end{cases},$$

where the subcycle  $\delta|_J$  is the ‘obstacle’ to forming  $\delta \boxplus \delta'$ , i.e. the subcycle of shape  $\delta'$  with  $m(\delta|_J) > m(\delta)$ . Conjugating  $\delta$  by  $\overline{\Omega}_J$  squeezes out that subcycle. By associativity, this defines the operator  $\boxplus_e$  defined on all of  $\Delta(\star) \times \Delta(\star)$ . ‘ $\boxplus_e$ ’ is an associative extension of ‘ $\boxplus$ ’: where  $\boxplus$  exists, it equals  $\boxplus_e$ . Although  $\Delta(n) \boxplus_e \Delta(n') = \Delta(n+n')$ , equation (3) will not always be satisfied. For example,  $(1)(26)(35)(4) \boxplus (13)(2)$  does not exist, but  $(1)(26)(35)(4) \boxplus_e (13)(2) = (1)(29)(38)(47)(56)$ . We will use  $\boxplus$  but not  $\boxplus_e$  in section 3.



### 3. Discussion

The monoidal structure ‘ $\boxplus$ ’ found in the previous section obeys (3), by construction, and so of course is ideally suited for enumerations involving cyclic structure in  $\Delta(n)$ . We give two examples.

When  $\delta_i \in \Delta(\star)$  are disjoint, i.e. don’t have any cycles with similar shapes, then both  $\delta_1 \boxplus \delta_2$  and  $\delta_2 \boxplus \delta_1$  will be defined. Hence we get equation (2).

In comparison with (2), the number of permutations in the symmetric group  $\mathfrak{S}_n$  which have precisely  $n_k$  disjoint subcycles of length  $k$  (so  $n = \sum n_k k$ ) is  $n! / \prod_k k \cdot n_k!$ .

Let  $\delta \in \Delta_k$ , and call  $\Delta_\delta(n)$  the set of all  $\delta' \in \Delta(n)$  possessing a subcycle of shape  $\delta$ : i.e.  $N_{\delta'}(\delta) > 0$ . Then  $\|\Delta_\delta(n)\| = (2^{n-k} - 2 \cdot \|\Delta_\delta(n-k)\|) + \|\Delta_\delta(n-k)\|$ , which can be solved to yield

$$\|\Delta_\delta(n)\| = \frac{1}{2^k + 1} (2^n - 2^\ell (-1)^{[n/k]}) , \quad (7)$$

where  $0 \leq \ell < k$  obeys  $n \equiv \ell \pmod{k}$ , and  $[x]$  is the greatest integer not more than  $x$ . Thus about  $\frac{2}{2^k + 1}$  of all unimodal permutations contain a given  $\delta \in \Delta_k$ .

By comparison, we find that the number of permutations in  $\mathfrak{S}_n$  which don’t possess *any*  $k$ -cycles (when written as a disjoint product of cycles) is precisely

$$n! \sum_{s=0}^{[n/k]} \left(-\frac{1}{k}\right)^s \frac{1}{s!} ,$$

and thus their density converges to  $e^{-\frac{1}{k}}$ .

Similar questions should be addressed for other pattern-avoiding sets  $\mathfrak{S}_n(\sigma, \sigma', \dots)$  of permutations. For example, any  $\pi \in \mathfrak{S}_n([231], [312])$  is an involution so is built from disjoint 1- and 2-cycles! Naturally, we can’t expect all such sets to be equally interesting from this perspective — e.g. no permutations for  $n > 4$  can avoid both patterns  $\{[123], [321]\}$ . The choice  $P = \{[123], [132]\}$  could be interesting to investigate from our point-of-view. Although there are  $2^{n-1}$  permutations which avoid  $P$ , as with  $\Delta(n)$ , there are two 3-cycles which avoid  $P$  (compared with  $\|\Delta_3\| = 1$ ), and both (14)(23) and (13)(24) have cycle structure  $(12) \oplus (12)$  (compared with only one unimodal sum  $(12) \oplus (12)$ ).

It is important here to consider the following symmetry. It is known [7] that there are 8 operations  $\mathfrak{S}_n \rightarrow \mathfrak{S}_n$ , forming the dihedral group  $\mathfrak{D}_4$ , that can be performed on our sets and which respect questions of pattern-avoidance. In particular, we can hit any permutation on the left or right with the involution  $\iota = [n, n-1, \dots, 1]$ , or we can replace a permutation by its inverse:  $\pi \mapsto \pi^{-1}$ . For any choice of operation  $\alpha \in \mathfrak{D}_4$ , the set  $\mathfrak{S}_n(\alpha(\sigma), \alpha(\sigma'), \dots)$  equals the set of all  $\alpha(\pi)$  for  $\pi \in \mathfrak{S}_n(\sigma, \sigma', \dots)$ . Half of these symmetries preserve in addition the cyclic structure: namely, the four operations  $\pi \mapsto \pi, \pi^{-1}, \iota \circ \pi \circ \iota, \iota \circ \pi^{-1} \circ \iota$ , which together form a  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry.

The unimodal permutations are precisely those which avoid both  $\{[213], [312]\}$ . Our  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry sends that to the sets  $\{[132], [231]\}$ ,  $\{132, [312]\}$  and  $\{[231], [213]\}$ , so their corresponding pattern-avoiding sets will also possess a monoidal structure and satisfy the same enumeration formulas (2),(7).

A question which seems to be relatively unexplored in the 1-dimensional dynamics literature is how distinct periodic orbits can nestle together in a given continuous map. This paper shows how severely constrained this is in the unimodal case. For instance, let  $c, c' \in \text{Int } I$  be the turning points of unimodal maps  $f, f' : I \rightarrow I$ . Let  $\mathcal{O}_i = \{m_i, f(m_i), f^2(m_i), \dots\}$ ,  $\mathcal{O}'_i = \{m'_i, f'(m'_i), \dots\}$  be sets of periodic orbits for  $f$  and  $f'$ , where  $m_i$  is the maximum point of  $\mathcal{O}_i$  (i.e.  $f(m_i) = \max \mathcal{O}_i$ ), and similarly for  $m'_i$ . It is a consequence of our work that *the finite bijections  $f|_{\cup_i \mathcal{O}_i}$  and  $f'|_{\cup_i \mathcal{O}'_i}$  will have identical permutation type*, if for each  $i$   $m_i$  and  $m'_i$  have the same ‘itinerary’ [4,3], i.e. (slightly more strongly) if for each  $i$ ,

- (i)  $\mathcal{O}_i$  and  $\mathcal{O}'_i$  correspond to the same cycle in  $\Delta_\star$ , and
- (ii) either  $m_i \leq c$  and  $m'_i \leq c'$ , or  $m_i \geq c$  and  $m'_i \geq c'$ .

For example, consider the nonconjugate maps  $f(x) = 0.939 \sin \pi x$  and  $f'(x) = 4x(1 - x)$ , and orbits  $\mathcal{O}_1 = \{0.5, .939, .179\}$ ,  $\mathcal{O}_2 = \{.376, .869\}$ ,  $\mathcal{O}'_1 = \{.611, .950, .188\}$ , and  $\mathcal{O}'_2 = \{.345, .905\}$ . Then the restrictions  $f|_{\mathcal{O}_1 \cup \mathcal{O}_2}$  and  $f'|_{\mathcal{O}'_1 \cup \mathcal{O}'_2}$  are both conjugate to the unimodal permutation (135)(24).

This observation can be regarded as a sort of combinatorial universality for unimodal functions. Condition (ii) is related to the fact that equation (2) involves a power of 2.

Consider now the logistic map  $x \mapsto 4x(1 - x)$ . All  $\delta \in \Delta_\star$  appear once or twice in it. A cycle will always appear there as a periodic orbit of ‘negative type’ or ‘orientation-reversing’ [8] (i.e. with their maximum  $< \frac{1}{2}$  for grave  $\delta$ , and  $> \frac{1}{2}$  for acute). Every  $\delta \in \Delta_n$  for odd  $n$  also appears as ‘positive type’, but for even  $n$  exactly  $\|\Delta_{n/2}\|$  (namely the doubles  $\mathcal{D}(\Delta_{n/2})$ ) do not appear as positive type (this is a consequence of [8]). For example, its fixed points are at  $x = 0$  (positive type) and  $x = \frac{3}{4}$  (negative); its unique 2-cycle is  $\{.345, .905\}$  (negative); and its 3-cycles are at  $\{.188, .611, .950\}$  (positive) and  $\{.117, .413, .970\}$  (negative). That quadratic map thus implies the existence (but not uniqueness) of many (but not all) sums  $\delta_1 \oplus \delta_2 \oplus \dots$ , and the ordering ‘ $\triangleleft$ ’ on  $\Delta_\star$  can be read off from it — e.g. since  $0 < .905 < .950$ , we have  $(1) \triangleleft (12) \triangleleft (123)$ .

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